Risk Measures with the CxLS property

Freddy Delbaen[1], Fabio Bellini[2], Valeria Bignozzi[3], Johanna Ziegel[4]

[1] *ETH Zürich and University of Zürich*, [2] *Università di Milano-Bicocca*, [3] *Università di Firenze*, [4] *University of Bern.*

Abstract

A law invariant risk measure ρ has the Convex Levels Sets property (CxLS) if

$$\rho(F) = \rho(G) = \gamma \Rightarrow \rho(\lambda F + (1-\lambda)G) = \gamma, \text{ for each } \gamma \in (0,1).$$

The level sets of ρ are convex with respect to mixtures of distributions; such a convexity has not to be confused with convexity or quasi-convexity with respect to sums of random variables. In the axiomatic theory of risk measures, the CxLS property arises naturally as a necessary condition for elicitability; a risk measure ρ is elicitable if it can be expressed as the minimizer of a suitable expected loss function $L\colon \mathbb{R}^2 \to \mathbb{R}$ as follows:

$$\rho(F) = \arg\min_{x\in\mathbb{R}} \int L(x,y)\, dF(y).$$

Moreover, the CxLS property has been introduced from a normative point of view in Decision Theory, where it plays the role of one of the various weakened versions of the Independence Axiom of the Von Neumann and Morgenstern theory.

In this work we characterize law invariant convex risk measures (that is, risk measures satisfying the properties of translation invariance, monotonicity and convexity with respect to sums) which also satisfy the CxLS property.

Our main result is that such risk measures are generalized shortfalls of the form

$$\rho(X) = \inf\{m \in \mathbb{R} \mid \int \ell(y-m)\, dF(y) \le 0\},$$

where $\ell\colon \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is increasing and convex with $\ell(-\infty) < 0 < \ell(+\infty)$.

The result is a variant of Weber's (2006) celebrated characterization of shortfall risk measures, also known as zero-utility premium principles in the actuarial literature.

As a byproduct of our analysis, we are able to prove a very simple characterization of robustness for convex risk measures. Indeed, we show that a law invariant convex risk measure ρ is Ψ-weak continuous for some gauge function Ψ, if and only if

$$\lim_{\lambda\to 0^+} \rho(\lambda\delta_x + (1-\lambda)\delta_y) = \rho(\delta_y), \text{ for each } x, y \in \mathbb{R}.$$