The comparison of smoothing methods in pension contracts

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June 27, 2019

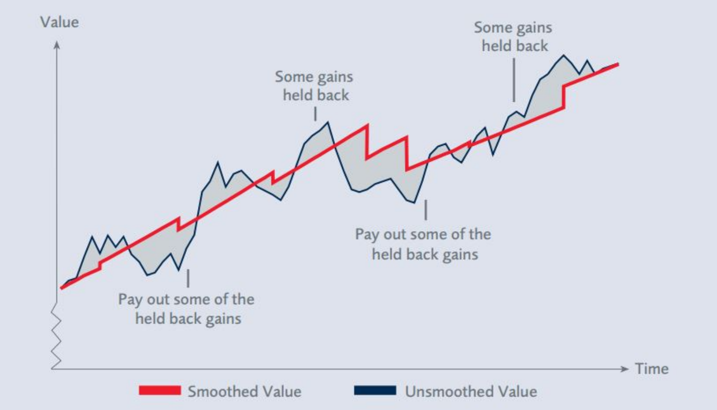
Background and Introduction

This paper compares three return smoothing methods of Life and Pension (L& P) products in UK.
The aims are:

- Examining if the smoothing methods provide a fair payout to the customers and understanding how the smoothing methods work.
- Identifying which smoothing method is more attractive to the customers.

What is smoothing?

- Smoothing method is used to smooth the extreme ups and downs of the market returns. The aim is expecting the value of customers' investment could cancel itself out over the long term.



Fairness

Three currently used smoothing methods in UK's L&P industry, geometric average (GA), weighted sum (WS) and Bandwidth (BW), are considered in this paper. We firstly examine if the smoothing methods generate a fair payout to the customers.
That is

$$E[S_N^{\chi}|\mathcal{F}_N] = E[S_N|\mathcal{F}_N] \qquad \text{for} \quad \chi \in \{GA, WS, BW\}. \tag{1}$$

where S_N^{χ} is the payout (smoothed terminal value of the investment) and S_N is the actual (unsmoothed) terminal value of the underlying investment.

Market model

A one-off premium P is paid at start and then invested in an investment fund for N years. The fund value is assumed to follow the geometric Brownian motion. Then the customer's investment S_t is

$$\begin{cases} S_0 &= P \\ dS_t &= \mu S_t dt + \sigma S_t dW_t^{\mathbb{P}}. \end{cases} \tag{2}$$

And the annual return is

$$R_n = \frac{S_n}{S_{n-1}} - 1 = \exp\left[\mu - \frac{1}{2}\sigma^2 + \sigma(W_n - W_{n-1})\right] - 1 \quad \text{for} \quad n = 1, 2 \ldots N. \tag{3}$$

Let $Y_n = 1 + R_n$, then $\{Y_n\}_{n=1}^N$ are independent copies of a log-normal distributed random variable Y with location $(\mu - \frac{1}{2}\sigma^2)$ and scale σ, i.e., $Y \sim \text{logN}(\mu - \frac{1}{2}\sigma^2, \sigma^2)$.

Smoothing method 1: Geometric average method

- The value of customer's payout is given as

$$S_n^{GA} = \begin{cases} P, & n = 0 \\ S_{n-1}^{GA}(1 + R_n^{GA}), & n \in \{1, 2, 3, ..., N\}. \end{cases} \quad (4)$$

where

$$R_n^{GA} = [(1 + R_{n-2})(1 + R_{n-1})(1 + R_n)(1 + R_{n+1}^*)(1 + R_{n+2}^*)]^{\frac{1}{5}} - 1 \quad (5)$$

Smoothing method 1: Geometric average method

For simplicity, we let $P = 1$. Then the expected smoothed value at the end of first year is

$$E[S_1^{GA}|\mathcal{F}_N] = E[(1+R_1^{GA})] = E\left[[(Y_{-1})(Y_0)(Y_1)(Y_2)(Y_3)]^{\frac{1}{5}}\right] = E[Y^{\frac{1}{5}}]^5 \tag{6}$$

Similarly, for year 2, we have

$$E[S_2^{GA}|\mathcal{F}_N] = E[(1+R_1^{GA})(1+R_2^{GA})] \tag{7}$$

$$= E\left[[(Y_{-1})(Y_0)(Y_1)(Y_2)(Y_3)]^{\frac{1}{5}}[(Y_0)(Y_1)(Y_2)(Y_3)(Y_4)]^{\frac{1}{5}}\right] = E[Y^{\frac{1}{5}}]^2 E[Y^{\frac{2}{5}}]^4$$

Smoothing method 1: Geometric average method

For the expected smoothed value from year 5 to year $N-2$,

$$E[S_n^{GA}|\mathcal{F}_N] = E[(1+R_1^{GA})(1+R_2^{GA})\cdots(1+R_n^{GA})] \tag{8}$$
$$= E\left[[(Y_{-1})(Y_0)(Y_1)(Y_2)(Y_3)]^{\frac{1}{5}}\cdots[(Y_{n-2})(Y_{n-1})(Y_n)(Y_{n+1})(Y_{n+2})]^{\frac{1}{5}}\right]$$
$$= E[Y^{\frac{1}{5}}]^2 E[Y^{\frac{2}{5}}]^2 E[Y^{\frac{3}{5}}]^2 E[Y^{\frac{4}{5}}]^2 E[Y]^{n-4}$$

For the terminal smoothed value

$$E[S_N^{GA}|\mathcal{F}_N] = E[(1+R_1^{GA})(1+R_2^{GA})\cdots(1+R_N^{GA})] \tag{9}$$
$$= E\left[[(Y_{-1})(Y_0)(Y_1)(Y_2)(Y_3)]^{\frac{1}{5}}\cdots[(Y_{N-2})(Y_{N-1})(Y_N)\exp(2\mu)]\right]$$
$$= E[Y^{\frac{1}{5}}]E[Y^{\frac{2}{5}}]E[Y^{\frac{3}{5}}]^2 E[Y^{\frac{4}{5}}]^2 E[Y_n]^{N-4}\exp\left(\frac{3\mu}{5}\right)$$

Smoothing method 1: Geometric average method

From Jensen's inequality, we know that

$$E[Y^p] < (E[Y])^p \quad \text{for} \quad 0 < p < 1. \tag{10}$$

Then the expected terminal smoothed value is

$$\begin{aligned} E[S_N^{GA}|\mathcal{F}_N] =& E[Y^{\frac{1}{5}}]E[Y^{\frac{2}{5}}]E[Y^{\frac{3}{5}}]^2E[Y^{\frac{4}{5}}]^2E[Y]^{N-4}\exp\left(\frac{3\mu}{5}\right) \\ <& (E[Y])^{\frac{1}{5}}(E[Y])^{\frac{2}{5}}(E[Y])^{\frac{6}{5}}(E[Y])^{\frac{8}{5}}(E[Y])^{N-4}\exp\left(\frac{3\mu}{5}\right) \\ =& (E[Y])^N = E[S_N] \end{aligned} \tag{11}$$

Alternatively, we are able to calculate all the moments of log-normal random variables. Thus,

$$E[S_N^{GA}|\mathcal{F}_N] = \exp\left(N\mu - \frac{3}{5}v^2\right) < \exp\left(N\mu\right) = E[S_N] \tag{12}$$

Smoothing method 1: Geometric average method

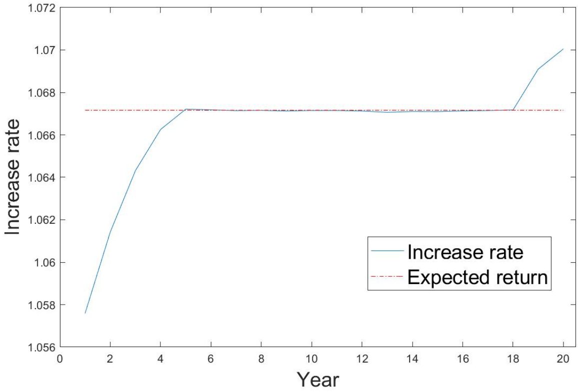


Figure: The increase rate $\frac{E[S_i^{GA}|\mathcal{F}_N]}{E[S_{i-1}^{GA}|\mathcal{F}_N]}$ of the expected value of the smoothed fund using GA method against the expected return of the actual fund value. $\mu = 0.065,\ \sigma = 0.15,\ N = 20,\ P = 1.$

Smoothing method 1: Geometric average method

Smoothing window (in years)	GA smoothed value
9	3.5781
7	3.5947
5	3.6195
3	3.6386
1 (no smoothing)	3.6670

Table: The effects of different smoothing windows on the expected terminal value under the GA method. $\mu = 0.065$, $\sigma = 0.15$, $N = 20$, $P = 1$.

Smoothing method 2: Weighted sum method

$$S_n^{WS} = \begin{cases} P, & n = 0 \\ \kappa S_{n-1}^{WS}(1+\mu') + (1-\kappa)S_n, & n \in \{1, 2, 3, ..., N\}. \end{cases} \quad (13)$$

where $\mu' = e^{\mu} - 1$ and $\kappa \in [0, 1]$ is a constant smoothing factor. After recursive substitution of Equation (13), we can get

$$S_N^{WS} = P\kappa^N(1+\mu')^N + (1-\kappa)\sum_{i=1}^{N} S_i \kappa^{N-i}(1+\mu')^{N-i} \quad (14)$$

Smoothing method 2: Weighted sum method

Taking the expectation gives

$$\begin{aligned} E[S_N^{WS}] &= P\kappa^N(1+\mu')^N + (1-\kappa)\sum_{i=1}^{N} E[S_i]\kappa^{N-i}(1+\mu')^{N-i} \\ &= P\kappa^N(1+\mu')^N + P(1-\kappa)\sum_{i=1}^{N}(E[Y])^i\kappa^{N-i}(1+\mu')^{N-i} \\ &= P\kappa^N(1+\mu')^N + P(1+\mu')^N(1-\kappa)\sum_{i=1}^{N}\kappa^{N-i} \\ &= P(1+\mu')^N = E[S_N] \end{aligned} \tag{15}$$

Smoothing method 3: BandWidth method

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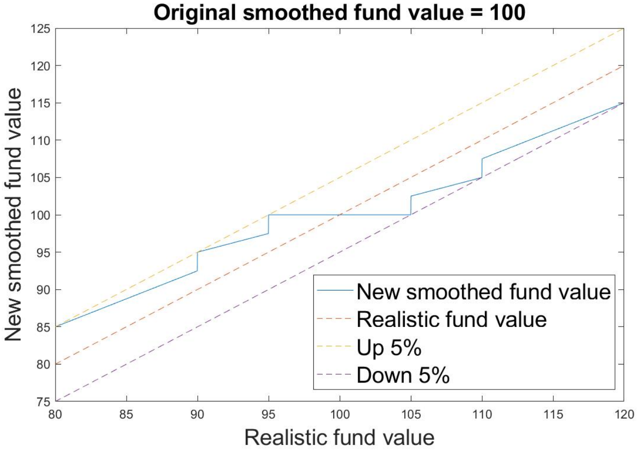


Figure: The smoothed fund value changes with actual fund value, when the original fund value equals 100.

Smoothing method 3: BandWidth method

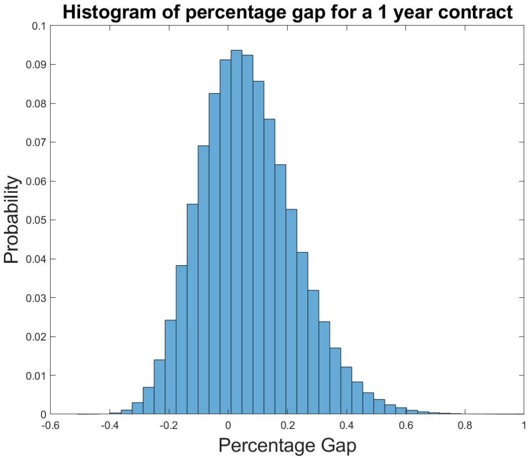


Figure: The histogram of the percentage gap based on the Monte Carlo simulation for a 1 year contract, under the BW method. The parameters are $\mu = 0.065$ and $\sigma = 0.15$.

Smoothing method 3: BandWidth method

The closed form solution to the expected value of the smoothed payout of BW method $E[S_N^{BW}]$ is not available. An approximation of S_1^{BW} can be expressed by $S_1^{BW^*}$.

$$S_1^{BW^*} = \begin{cases} S_1 - \frac{1}{8}[S_1 - Pe^{\mu}] = \frac{7}{8}S_1 + \frac{1}{8}Pe^{\mu}, \ \text{for } S_1 \in (1.2Pe^{\mu}, +\infty) \\ S_1 - \frac{1}{4}[S_1 - Pe^{\mu}] = \frac{3}{4}S_1 + \frac{1}{4}Pe^{\mu}, \ \text{for } S_1 \in (1.1Pe^{\mu}, 1.2Pe^{\mu}] \\ S_1 - \frac{1}{2}[S_1 - Pe^{\mu}] = \frac{1}{2}S_1 + \frac{1}{2}Pe^{\mu}, \ \text{for } S_1 \in (1.05Pe^{\mu}, 1.1Pe^{\mu}] \\ Pe^{\mu}, \ \text{for } S_1 \in (0.95Pe^{\mu}, 1.05Pe^{\mu}] \\ S_1 + \frac{1}{2}[Pe^{\mu} - S_1] = \frac{1}{2}S_1 + \frac{1}{2}Pe^{\mu}, \ \text{for } S_1 \in (0.9Pe^{\mu}, 0.95Pe^{\mu}] \\ S_1 + \frac{1}{4}[Pe^{\mu} - S_1] = \frac{3}{4}S_1 + \frac{1}{4}Pe^{\mu}, \ \text{for } S_1 \in (0.8Pe^{\mu}, 0.9Pe^{\mu}] \\ S_1 + \frac{1}{8}[Pe^{\mu} - S_1] = \frac{7}{8}S_1 + \frac{1}{8}Pe^{\mu}, \ \text{for } S_1 \in (0, 0.8Pe^{\mu}]. \end{cases} \tag{16}$$

By deriving the above equation, we have

$$E[S_1^{BW}] > E[S_1^{BW^*}] > E[S_1]. \tag{17}$$

Smoothing method 3: BandWidth method

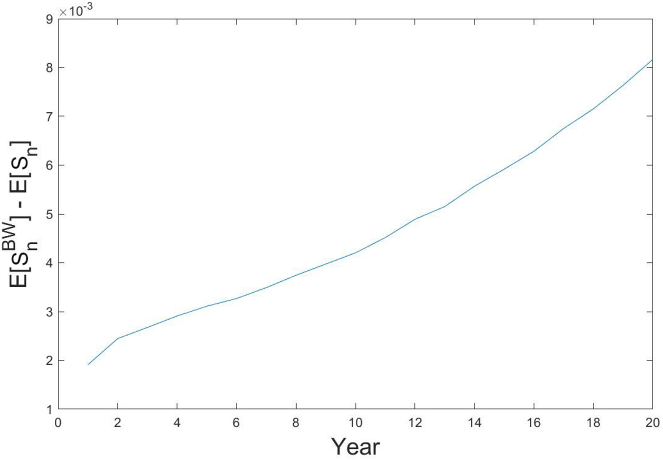


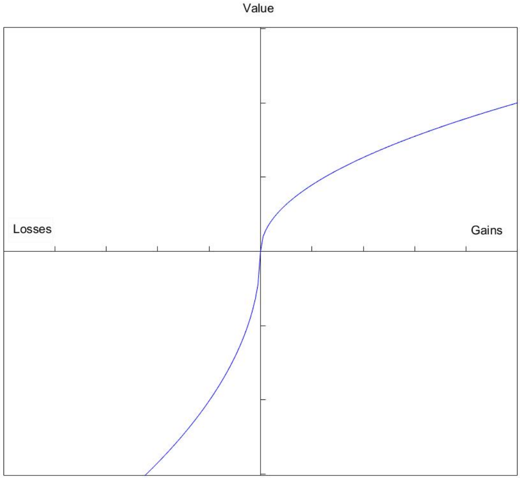
Figure: The difference between the expected value using BW smoothing method and the unsmoothed value , $E[S_n^{BW}] - E[S_n]$. The parameters are $\mu = 0.065$, $\sigma = 0.15$, $N = 20$, $P = 1$.

Utility calculation

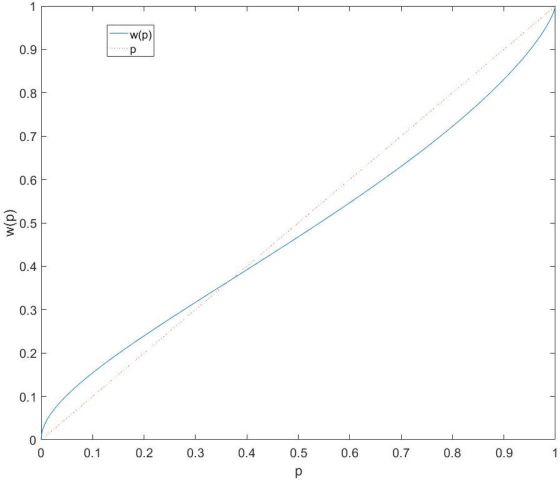
How people compare different smoothing methods(investments)?

- Expected Utility Theory
- Cumulative Prospect Theory
  Tversky and Kahneman (1992) proposed the Cumulative Prospect Theory (CPT) to explain some violations of EUT.
- Multi-Cumulative Prospect Theory
  Ruß and Schelling (2018) propose the Multi-Cumulative Prospect Theory to consider the utility of the changes in value of investment within the investment horizon.

Value Function



Weighting function



Cumulative Prospect Theory

- Model:

Let f denote a prospect $(x_{-m}, p_{-m}; x_{-m+1}, p_{-m+1}; ...; x_0, p_0; ...; x_{n-1}, p_{n-1}; x_n, p_n;)$ where x_i is the outcome and p_i is the corresponding probability. In CPT, outcomes are arranged in ascending order. CPT values gains and loss separately. The utility of a prospect is the sum of utility of positive prospect f^+ and negative prospect f^-. The formula is given as:

$$V(f) = V(f^+) + V(f^-) = \sum_{i=0}^{n} \pi_i^+ v(x_i) + \sum_{i=-m}^{0} \pi_i^- v(x_i) \tag{18}$$

where v is the value function and π_i is the decision weights associated with outcome x_i.

Multi-Cumulative Prospect Theory

Let X_t denote the value change of the investor's payout in year t, then the MCPT utility of an investment A is given by

$$CPT^{com}(A) := sMCPT(A) + (1-s)CPT(A) \tag{19}$$

$$= s\sum_{t=1}^{T} V(X_t) + (1-s)V(S_T^{\chi} - S_0) \tag{20}$$

Numerical result

In addition to the GBM model, we use the Bi-variate trending Ornstein-Uhlenbeck(OU) process to study how the smoothing methods perform when the market returns are correlated. The bivariate trending OU process is defined as

$$\begin{cases} dS_t = \Big(\mu - \kappa\big(\log\frac{S_t}{S_0} - (\mu - \frac{1}{2}\sigma^2)t\big) + \lambda H_t\Big) S_t dt + \sigma S_t dW_t^{(s)}, \\ dH_t = -\delta H_t dt + \sigma_x dW_t^{(h)}, \\ S_0 = s, H_0 = h, \end{cases} \tag{21}$$

where $W_t^{(s)}$ and $W_t^{(h)}$ are two independent standard Brownian motions, $\kappa \geq 0$ and $\delta \geq 0$ are the mean reverting parameters of processes S_t and H_t,respectively.

GBM	Actual	GA	WS	BW
mean of terminal value	6.051	5.931	6.050	6.064
variance of terminal value	35.02	29.55	31.17	34.29
CPT^{com}	0.7253	3.5081	2.8066	1.4830

BiVar Trend	Actual	GA	WS	BW
mean of terminal value	6.186	6.105	6.186	6.199
variance of terminal value	1.77	0.70	1.25	1.58
CPT^{com}	1.0667	5.3651	3.9640	2.0974

Table: Mean, variance and CPT^{com} of the terminal payout of a 20 years contract from different smoothing methods.

Thank You!

Questions Time