Idempotents of the Hecke algebra become Schur functions
in the skein of the annulus

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Abstract

The Hecke algebra $H_n$ can be described as the skein $R^n_n$ of $(n, n)$-tangle diagrams with respect to the framed Homfly relations. This algebra $R^n_n$ contains well-known idempotents $E_\lambda$ which are indexed by Young diagrams $\lambda$ with $n$ cells. The geometric closures $Q_\lambda$ of the $E_\lambda$ in the skein of the annulus are known to satisfy a combinatorial multiplication rule which is identical to the Littlewood–Richardson rule for Schur functions in the ring of symmetric functions. This fact is derived from two skein theoretic lemmas by using elementary determinantal arguments. Previously known proofs depended on results for quantum groups.

1. Introduction

Sections 2 and 3 provide an overview on skeins, Young diagrams and symmetric functions. In Section 4 the skein $R^n_n$ of $(n, n)$-tangles is presented. $R^n_n$ contains idempotents $E_\lambda$ indexed by Young diagrams $\lambda$ with $n$ cells. These idempotents were described algebraically by Gyoja [5] and were presented in a skein theoretic formulation by Yokota [14] and by Aiston and Morton [1, 2].

The main ideas start in Section 5 where the skein $C$ of the annulus which has a commutative multiplicative structure is introduced. The canonical closure operation induces a map from $R^n_n$ to $C$ and the closure of $E_\lambda$ is denoted by $Q_\lambda$.

The skein $C$ of the annulus is well known from the calculation of invariants of satellite knots because the framed Homfly polynomial of a satellite knot depends (beside the companion knot) only on the skein class of the decoration. The skein elements $Q_\lambda$ are a basis of the important subring $C_+$ of skein elements whose components have a strictly counter-clockwise orientation. The understanding of the multiplication rule for the $Q_\lambda$ is therefore of great interest. For example, it allows in [11] the computation of the framed Homfly polynomial of the decorated Hopf link. The substitution of the variables of these polynomials by appropriate roots of unity leads to the realm of 3-manifold invariants (see e.g. [1, 3, 7, 9, 14] for more details and references).

In Section 6, I consider the homomorphism from the ring of symmetric functions to $C$ which maps the $i$th complete symmetric function $h_i$ to $Q^{(i)}$ where $(i)$ denotes a row diagram with $i$ cells. The homomorphic image of the Schur function $s_\lambda$ is denoted by $S_\lambda$ and is expressed by an explicit determinantal formula. Only in Theorem 10-1
will I be able to prove that $S_\lambda$ is equal to $Q_\lambda$ for any Young diagram $\lambda$ which immediately implies the main Theorem 10.3 that the multiplication of the $Q_\lambda$ follows the Littlewood–Richardson rule for Schur functions. This was proved first by Aiston [1, theorem 4.8.8] using quantum groups.

The idea for proving the equality of $S_\lambda$ and $Q_\lambda$ is to construct a linear map $\Gamma : \mathbb{C}_+ \to \mathbb{C}_+$ in such a way that the skein elements $S_\lambda$ and $Q_\lambda$ are eigenvectors of $\Gamma$ and their eigenvalues are equal for any Young diagram $\lambda$. If the eigenvalues for all the $Q_\lambda$ are pairwise different then $S_\lambda$ is a scalar multiple of $Q_\lambda$. The linear map $\Gamma$ used in this paper is the encirclement with a single loop (following Kawagoe [6]). $Q_\lambda$ is an eigenvector of $\Gamma$ because the identity braid with an encircling loop is a central element of $R_n^0$ and is therefore swallowed by any $E_\lambda$ at the expense of a scalar [2, lemma 5.2]. These scalars are the eigenvalues and they are pairwise different.

The variant skein $\mathcal{C}'$ of the annulus with two boundary points is defined in Section 7. This skein was introduced by Morton [10] and was shown by him to be a commutative ring. It is the appropriate setting for proving the skein theoretic Lemmas 8.3 and 8.4 in Section 8. In Section 9, I deduce from these two lemmas by purely determinantal calculations that $S_\lambda$ is an eigenvector of the map $\Gamma$ and that the eigenvalues for $S_\lambda$ and $Q_\lambda$ are equal for any Young diagram $\lambda$. As mentioned before, it follows that $S_\lambda$ is a scalar multiple of $Q_\lambda$.

In the proof of [1, theorem 4.8.8] Aiston stated that, after the inclusion of the annulus into the plane, the framed Homfly polynomials of $S_\lambda$ (in her notation $\theta(\lambda)$) and $Q_\lambda$ are equal and non-zero. Since she used results for quantum groups I cannot follow this direct route to prove the equality of $S_\lambda$ and $Q_\lambda$. Instead, an explicit semi-simple decomposition of $R_n^0$ described by Blanchet [3] gives enough information to conclude in Section 10 that $S_\lambda$ is equal to $Q_\lambda$ for any Young diagram $\lambda$.

2. The skein of a planar surface

Let $F$ be an oriented planar surface with designated $n$ incoming and $n$ outgoing boundary points for some integer $n \geq 0$. A tangle $T$ in $F$ is a regular diagram that consists of oriented closed curves and $n$ oriented arcs whose boundary points agree with the designated boundary points of $F$. The arcs of $T$ are oriented towards the incoming boundary points of $F$. The orientation of the closed components of $T$ is not restricted. We remark that a tangle as we use it is usually called a tangle diagram in the literature.

The skein $S(F)$ is defined as the module of linear combinations of tangles in $F$ quotiented by regular isotopy (i.e. Reidemeister moves II and III), the two local relations in Figure 1 where $v$ and $s$ are variables, and the relation that a null-homotopic disjoint simple closed curve can be removed from a tangle at the expense of multiplication with the scalar $\delta = (v^{-1} - v)/(s - s^{-1})$. This last relation for removing a circle is a consequence of the other two relations except when the remaining tangle is the empty tangle. We remark that the relation for removing a positive curl in Figure 1 has to be required for both of the two possible orientations.

The set of scalars for a skein can be chosen as $\mathbb{Z}[v^\pm 1, s^\pm 1]$ with powers of $s - s^{-1}$ in the denominators. In Section 4 we have to divide a quasi-idempotent by a scalar to turn it into an idempotent. Therefore, we shall consider throughout this paper the field of rational functions in $v$ and $s$ as the set of scalars.
For the skein of any oriented planar surface \( F \) we define a map \( \rho: S(F) \to S(F) \). It is induced by switching all crossings of a tangle and replacing \( s \) by \( s^{-1} \) and \( v \) by \( v^{-1} \) which preserves the skein relations. We remark that \( \rho^2 = \text{id} \). The map \( \rho \) is additive but not linear because it changes the scalars.

It is possible to define an apparently more general version of the skein relations by disturbing the skein relations with an additional variable \( x \) (see e.g. [7, section 2.1] for details). In the case \( x = 1 \) this agrees with our version. In fact, the two skeins are always isomorphic by mapping \( D \mapsto x^{-\text{wr}(D)} D \) where \( \text{wr}(D) \) is the writhe of a tangle \( D \), i.e. the weighted sum of positive and negative crossings of \( D \).

### 3. Young diagrams and symmetric functions

#### 3.1. Young diagrams

A partition of a non-negative integer \( n \) is a non-strictly decreasing sequence of non-negative integers that add up to \( n \). We consider partitions to be equal if they differ only by some zeros at the end.

A Young diagram is a graphical depiction of a partition by square cells. A partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \) is displayed as a flush-left arrangement of square cells with \( \lambda_i \) cells in the \( i \)th row for each \( 1 \leq i \leq k \). The rows are numbered top-down.

We define a partial ordering for partitions \( \lambda = (\lambda_1, \ldots, \lambda_k) \) and \( \mu = (\mu_1, \ldots, \mu_m) \) by saying that \( \lambda \subset \mu \) if \( \lambda_i \leq \mu_i \) for each \( 1 \leq i \leq k \) where \( \mu_j = 0 \) for \( m + 1 \leq j \leq k \). This ordering coincides in the graphical depiction with \( \lambda \) being a subset of \( \mu \).

We define \( |\lambda| \) to be the number of cells of \( \lambda \), i.e. for \( \lambda = (\lambda_1, \ldots, \lambda_k) \) we have \( |\lambda| = \lambda_1 + \ldots + \lambda_k \).

The length \( l(\lambda) \) of a Young diagram \( \lambda \) is the number of non-zero rows of \( \lambda \).

#### 3.2. Symmetric functions

Following the exposition of Macdonald [8], we consider the polynomial ring in \( n \) variables \( x_1, \ldots, x_n \). For a partition \( \lambda = (\lambda_1, \ldots, \lambda_k) \) with \( k \leq n \) we define

\[
s_\lambda(x_1, \ldots, x_n) = \frac{\det(x_i^{\lambda_j + n - j})_{1 \leq i,j \leq n}}{\det(x_i^{n - j})_{1 \leq i,j \leq n}}.
\]

The two determinants in this fraction are skew-symmetric polynomials and therefore the quotient is a symmetric polynomial in \( x_1, \ldots, x_n \). It is a polynomial, indeed. In the inverse limit of the polynomial rings \( \mathbb{Z}[x_1, \ldots, x_n] \) for \( n \to \infty \) one can define an element \( s_\lambda \), called the \( \lambda \)-Schur function, for any Young diagram \( \lambda \). This element specialises to \( s_\lambda(x_1, \ldots, x_n) \) when we set \( x_i = 0 \) for all \( i \geq n + 1 \) provided that \( n \) is greater than or equal to the length of \( \lambda \). The set of Schur functions \( s_\lambda \) for all Young diagrams \( \lambda \) is a \( \mathbb{Z} \)-basis of the ring of symmetric functions.

The most basic Young diagrams are on the one hand a row diagram with, say, \( n \) cells and on the other hand a column diagram with, say, \( k \) cells. One can check that
s_{(n)} is the $n$th complete symmetric function $h_n$ and that $s_{(i^k)}$ is the $k$th elementary symmetric function $e_k$.

The Littlewood–Richardson rule from [8] states that

$$s_\lambda s_\mu = \sum |\nu| = |\lambda| + |\mu| a_{\lambda\mu}^\nu s_\nu$$

with integers $a_{\lambda\mu}^\nu \geq 0$ for any Young diagrams $\lambda$ and $\mu$.

4. The skein $R_n^n$ of a disk with $2n$ boundary points

We consider a rectangle $F$ with $n$ outgoing points on the top edge and $n$ incoming points on the bottom edge, $n \geq 0$. We denote the skein of this surface by $R_n^n$.

$R_n^n$ gets an algebra structure by defining the product $D \cdot E$ as putting the tangle $D$ over the tangle $E$ so that the incoming points of $D$ match with the outgoing points of $E$. Clearly, $R_n^n$ is generated as an algebra by the elementary braids $\sigma_1, \ldots, \sigma_{n-1}$ where $\sigma_i$ agrees with the identity braid up to a single positive crossing connecting the boundary points $i$ and $i + 1$ (numbered from left to right). $R_n^n$ is known to be isomorphic to the Hecke algebra $H_n$.

As a module, $R_n^n$ is spanned linearly by the positive permutation braids $\omega_\pi$ on $n$ strings where $\pi$ runs through the permutations on $n$ letters (see [9] or [7, section 2·3] for details).

$R_n^n$ has a well known quasi-idempotent $a_n$ which is given in terms of the positive permutation braids as

$$a_n = \sum_{\pi \in S_n} s^{l(\pi)} \omega_\pi$$

where $l(\pi)$ is the minimal number of transpositions to represent $\pi$. This integer $l(\pi)$ is equal to the writhe of the braid $\omega_\pi$ as defined at the end of Section 2. We understand $a_0$ to be the empty tangle in $R_0^0$. This special case needs some additional, but trivial, arguments in this section.

It turns out that the element $a_n$ ‘swallows’ an elementary braid $\sigma_i$ at the expense of the scalar $s$ as stated in the following lemma which is proved in [9].

**Lemma 4·1.** $a_n \sigma_i = \sigma_i a_n = sa_n$ for any integers $n$ and $i$ with $n \geq i + 1 \geq 2$.

This Lemma implies that $a_n$ is central in $R_n^n$ because the elementary braids $\sigma_1, \ldots, \sigma_{n-1}$ generate $R_n^n$. The lemma also implies that $a_n a_n = \alpha_n a_n$ for a scalar $\alpha_n$ for any integer $n \geq 1$. This scalar is non-zero and can be computed inductively as

$$\alpha_n = s^{n(n-1)} [n][n-1] \cdots [1]$$

where the quantum integer $i$ is defined by $[i] = (s^i - s^{-i})/(s - s^{-1})$ for any integer $i \geq 1$ (see e.g. [7, lemma 2·4·2]). Hence, the element $(1/\alpha_n)a_n$ is an idempotent of $R_n^n$ for any integer $n \geq 1$. It is invariant under the map $\rho$ from Section 2. The same holds for $n = 0$ when we define $\alpha_0 = 0$.

**Lemma 4·2.** $\rho((1/\alpha_n)a_n) = (1/\alpha_n)a_n$ for any integer $n \geq 0$.

**Proof.** The map $\rho: R_n^n \to R_n^n$ is a ring homomorphism but not an algebra homomorphism because it changes the scalars. Nevertheless, the multiplicativity
\[ \rho(fg) = \rho(f)\rho(g) \] for any elements \( f, g \in R_n^\varepsilon \) together with Lemma 4.1 implies that
\[ \sigma_i \rho(a_n) = \rho(\sigma_i^{-1})\rho(a_n) = \rho(\sigma_i^{-1}a_n) = \rho(s^{-1}a_n) = s\rho(a_n) \]
in \( R_n^\varepsilon \). The equation \( \sigma_i \rho(a_n) = s\rho(a_n) \) implies that \( a_n \rho(a_n) = \alpha_n \rho(a_n) \) for the same scalar \( \alpha_n \) as above. When we apply the map \( \rho \) to this equation we get \( \rho(a_n)\rho(a_n) = \rho(\alpha_n)a_n \). The left-hand sides of the previous two equations are equal because \( a_n \) is central. Hence, \( \alpha_n \rho(a_n) = \rho(\alpha_n)a_n \) and thus \( \rho(1/\alpha_n)a_n = (1/\alpha_n)a_n \).

Aiston and Morton [2] constructed idempotents \( E_\lambda \) in \( R_n^\varepsilon \) for any Young diagram \( \lambda \) with \( n \) cells. To do this, they considered a three-dimensional version of \( R_n^\varepsilon \) where the distinguished boundary points are lined up along the cells of the Young diagram \( \lambda \). In this setting, Aiston and Morton arranged idempotents \( a_{\lambda_i} \) along the rows of \( \lambda \) and variant idempotents \( b_{\lambda^\gamma} \) along the columns of \( \lambda \). Here, \( \lambda^\gamma \) derives from \( \lambda \) by transposing, i.e. interchanging columns and rows. The resulting skein element is denoted by \( e_\lambda \) in the equivalent standard two-dimensional \( R_n^\varepsilon \)-setting where it satisfies \( e_\lambda e_\lambda = \alpha_\lambda e_\lambda \) for a scalar \( \alpha_\lambda \). In order to prove that \( e_\lambda \neq 0 \) and \( \alpha_\lambda \neq 0 \) it is sufficient to prove it for the well known specialisation \( v = s = 1 \) which turns \( R_n^\varepsilon \) into the symmetric-group algebra (see [1]).

The element \( E_\lambda = (1/\alpha_\lambda)e_\lambda \) is an idempotent of \( R_n^\varepsilon \). In particular, we have \( E_{(n)} = (1/\alpha_n)a_n \) for any integer \( n \geq 0 \). In this paper we consider the whole ring of rational functions in \( v \) and \( s \) as the set of scalars. One has to be careful about the scalars e.g. in the context of 3-manifold invariants. Then, the variables \( v \) and \( s \) are substituted by some roots of unity and the denominators might become zero (see e.g. [1] and [7]).

We indicate a second proof that \( e_\lambda \neq 0 \) in \( R_n^\varepsilon \). We consider the inclusion of the closure of \( e_\lambda \) into the skein of the plane. We make the specialisation \( v = s = 1 \) and take care of \( \delta \) by keeping it as a variable. Over- and undercrossings become interchangeable and the closure of \( e_\lambda \) becomes the empty tangle multiplied with a polynomial in \( \delta \). The coefficient of \( \delta^\alpha \) can be shown to be equal to 1 which proves that the closure of \( e_\lambda \) is non-zero [7, lemma 2.4.6].

5. The skein \( \mathcal{C} \) of the annulus

The skein of the annulus is denoted by \( \mathcal{C} \). Let \( D_1 \) and \( D_2 \) be tangles in the annulus \( S^1 \times [0, 1] \). We can bring \( D_1 \) into \( S^1 \times (0, \frac{1}{2}) \) and \( D_2 \) into \( S^1 \times (\frac{1}{2}, 1) \) by regular isotopies. Then the product of \( D_1 \) and \( D_2 \) is defined as the tangle \( D_1 \cup D_2 \). An example is shown in Figure 2. The multiplication is commutative because \( D_1 \cdot D_2 \) and \( D_2 \cdot D_1 \) differ by a regular isotopy. The skein \( \mathcal{C} \) is therefore a commutative ring with the empty tangle.
as the identity element. Furthermore, the skein $\mathcal{C}$ can be regarded as a commutative algebra over our chosen ring of scalars, namely the rational functions in $v$ and $s$.

Figure 3 depicts an annulus with a set of $n$ oriented arcs. A rectangle is removed from the annulus so that we can insert a tangle from $R_n^n$. This factors to a map $\Delta_n: R_n^n \to \mathcal{C}$ for any $n \geq 0$. This is a special case of a wiring. We abbreviate $\Delta_n$ by $\Delta$ if the index $n$ is obvious from the context.

We denote by $\mathcal{C}_n$ the image of $R_n^n$ under the closure map $\Delta_n$. It is a submodule of $\mathcal{C}$. In particular, $\mathcal{C}_0$ is the submodule of scalar multiples of the empty tangle. We have $\mathcal{C}_n \cdot \mathcal{C}_m \subset \mathcal{C}_{n+m}$ for any integers $n, m \geq 0$.

By $\mathcal{C}_+$ we denote the submodule of $\mathcal{C}$ spanned by the set of $\mathcal{C}_i$ for all $i \geq 0$.

We define $Q_\lambda$ to be the closure of the idempotent $E_\lambda$ of $R_n^n$ where $n$ is the number of cells of $\lambda$,

$$Q_\lambda = \Delta_n(E_\lambda) \in \mathcal{C}_n.$$  

At the end of Section 4 we remarked that the closure of $e_\lambda$ as an element of the skein of the plane is non-zero. Hence, the inclusion of $Q_\lambda$ into the skein of the plane gives a non-zero element which implies that $Q_\lambda$ is non-zero in $\mathcal{C}$.

We define a linear map $\Gamma: \mathcal{C} \to \mathcal{C}$ that encircles a tangle in the annulus with a single loop which has a specified orientation as shown in Figure 4. It is clear that $\Gamma(\mathcal{C}_i) \subset \mathcal{C}_i$ for any integer $i \geq 0$ and thus $\Gamma(\mathcal{C}_+) \subset \mathcal{C}_+$.

In $R_n^n$, the identity braid on $n$ strings with an encircling loop can be written as a linear combination of Jucys–Murphy operators as explained in [10]. Each of these summands is swallowed by $e_\lambda$ at the expense of a scalar which is computed skein theoretically in [2]. Summing up these scalars immediately implies the following lemma. For details see [7, lemma 2.4.7].
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**Lemma 5.1.** For any Young diagram $\lambda$ we have $\Gamma(Q_{\lambda}) = c_{\lambda}Q_{\lambda}$ in $C_+$ with the scalar

$$c_{\lambda} = \frac{v^{-1} - v}{s - s^{-1}} + vs^{-1} \sum_{k=1}^{l(\lambda)} (s^{2k-\lambda_k} - s^{2k}).$$

**Remark.** It is easy to confirm that the eigenvalues $c_{\lambda}$ are pairwise different.

**Lemma 5.2.** The set $\{Q_{\lambda} \mid \lambda \text{ has } n \text{ cells}\}$ is a basis of $C_n$ for any integer $n \geq 0$.

**Proof.** We define $X_i^+ \in C_i$ as the closure of the braid $\sigma_{i-1}\sigma_{i-2}\cdots\sigma_1 \in R_i^+$. The **weighted degree** of a monomial $(X_i^+)^{j_1} \cdots (X_{i_k}^+)^{j_k}$ is defined as $i_1j_1 + \cdots + i_kj_k$.

Any tangle in the annulus can be written inductively via the skein relations as a linear combination of tangles each of which is totally descending. This means that $C_n$ is spanned linearly by the monomials in $X_1^+, \ldots, X_n^+$ of weighted degree $n$. Hence, the dimension of $C_n$ is at most $\pi(n)$, by which we denote the number of partitions of $n$.

On the other hand, any $Q_{\lambda}$ lies in $C_n$ provided that the Young diagram $\lambda$ has $n$ cells. The $Q_{\lambda}$ are non-zero and they are linearly independent because they have pairwise different eigenvalues with respect to the map $\Gamma$ by Lemma 5.1. By definition, there are $\pi(n)$ Young diagrams with $n$ cells. Hence, there are at least $\pi(n)$ linearly independent elements in $C_n$. Since the dimension of $C_n$ is at most $\pi(n)$ by the above argument, the dimension of $C_n$ is exactly $\pi(n)$ and the set of $Q_{\lambda}$ where $\lambda$ has $n$ cells is a basis.

**Corollary 5.3.** The set $\{Q_{\lambda} \mid \text{all Young diagrams } \lambda\}$ is a basis of $C_+$.

**Proof.** $C_+$ consists of linear combinations of elements of $C_n$, $n \geq 0$. It follows from Lemma 5.2 that $C_+$ is spanned by the set $\{Q_{\lambda} \mid \text{all Young diagrams } \lambda\}$. The $Q_{\lambda}$ are linearly independent because they are non-zero and they have pairwise different eigenvalues with respect to the map $\Gamma$.

**Remark.** It follows from the proof of Lemma 5.2 and from Corollary 5.3 that the set of elements $X_i^+$ for all $i \geq 0$ generates $C_+$ freely as a commutative algebra. This is a special case of Turaev’s result [12] that the whole skein $\mathcal{C}$ is generated freely as a commutative algebra by the set of $X_i^+$ and $X_i^-$ for all $i \geq 0$ where $X_i^-$ derives from $X_i^+$ by reversing the orientation.

6. The skein $\mathcal{C}$ of the annulus and symmetric functions

We recall that $(i)$ denotes the row diagram with $i$ cells. We denote the skein element $Q_{(i)} \in C_+$ by $h_i$ for any integer $i \geq 0$. Since $E_{(i)} = (1/\alpha_i)a_i$, we have $h_i = (1/\alpha_i)\Delta(a_i)$ for any integer $i \geq 0$. We define $h_i = 0 \in C_+$ for any $i \leq -1$.

The ring of symmetric functions (in countably many variables) is freely generated as a commutative algebra by the set of complete symmetric functions $h_i$ for all $i \geq 0$. For any $i \leq -1$ we set $h_i = 0$. The multiple use of the letter $h$ is an abuse of notation but by considering the context it is possible to distinguish between a complete symmetric function $h_i$ and a skein element $h_i$.

We induce a well-defined ring homomorphism from the ring of symmetric functions to the skein of the annulus by mapping the complete symmetric function $h_i$ to the skein element $h_i \in \mathcal{C}$. We denote the image of the Schur function $s_\lambda$ under this map
by $S_{\lambda}$. The Schur function $s_{\lambda}$ can be expressed in terms of the complete symmetric functions via the formula $s_{\lambda} = \det (h_{\lambda, j-i})_{1 \leq i, j \leq l(\lambda)}$ from [8]. Therefore, we have

$$S_{\lambda} = \det (h_{\lambda, j-i})_{1 \leq i, j \leq l(\lambda)} \in \mathcal{C}_n$$

where $n = |\lambda|$. We deduce that $S_{(i)} = h_i$ for any integer $i \geq 0$. Hence, $S_{\lambda} = Q_{\lambda}$ for any row diagram $\lambda$.

Only in Theorem 10.1 will we be able to prove that $S_{\lambda} = Q_{\lambda}$ for any Young diagram $\lambda$. We remark that Aiston [1, theorem 4·8·4] gave a skein theoretic proof that $S_{\lambda} = Q_{\lambda}$ for any hook-shaped Young diagram $\lambda$, i.e. $\lambda = (a, 1^b)$ for positive integers $a$ and $b$.

It follows from the above determinantal formula for $S_{\lambda}$ and from the formula for the scalars $\alpha_i$ from Section 4 that $S_{\lambda}$ can be written as a linear combination of tangles in such a way that the denominators of the scalars are products of quantum integers $[i]$ where $1 \leq i \leq \lambda_1 + l(\lambda) - 1$. The upper bound $\lambda_1 + l(\lambda) - 1$ is also known as the hook length of $\lambda$.

It is possible to prove at this stage that $S_{\lambda}$ is non-zero in $\mathcal{C}$ for any Young diagram $\lambda$. To do this, one considers the inclusion of the annulus into the plane. The inclusion maps $S_{\lambda}$ and $h_i$ to skein elements $\tilde{S}_{\lambda}$ and $\tilde{h}_i$, respectively. The framed Homfly polynomial of $\tilde{h}_i$ is well known and the framed Homfly polynomial of $\tilde{S}_{\lambda}$ is the $\lambda$-Schur function of these values which is quickly shown to be non-zero (see [7, lemma 3·6·1]). In what follows we do not need this observation.

7. The variant skein $\mathcal{C}'$ of the annulus

The skein of the annulus $\mathcal{C}'$ with two boundary points appeared e.g. in [4] and [6]. We are using the version introduced by Morton in [10] apart from a turn over of the annulus. His version has the benefit of a commutative multiplication as we explain now.

We equip the annulus with two distinguished boundary points. An output point on the inner circle and an input point on the outer circle. The resulting skein $\mathcal{C}'$ becomes an algebra by defining the product $A \cdot B$ of tangles $A$ and $B$ as putting the inward circle of $A$ next to the outward circle of the shrunk tangle $B$ so that the two involved distinguished boundary points become a single point in the interior of the new annulus. An example for the multiplication is depicted in Figure 5. The single straight arc $e$ that connects the two marked points as shown in Figure 6 is the identity element.

Any arc that connects the two marked points without crossing itself is regularly isotopic to an integer power of the tangle $a$ which is depicted in Figure 7.
The tangles $A \cdot B$ and $B \cdot A$ are not regularly isotopic in general but they are equal modulo the skein relations as will be shown in Lemma 7.1.

We note that $C'$ can be turned into an algebra over $C$ in two ways. Let $A$ be a tangle in $C'$ and $\gamma$ a tangle in $C$. Then $\gamma A$ is defined as putting $\gamma$ over $A$, and $A \gamma$ is defined as putting $\gamma$ under $A$. For example, Figure 5 reads $a^{-2} \cdot (h_1 a) = h_1 a^{-1}$.

Morton uses in [10] the notation $l(\gamma, A)$ and $r(A, \gamma)$ for the operation of $C$ on $C'$ on the left and on the right, respectively.

**Lemma 7.1 ([10]).** $C'$ is commutative.

**Proof.** Let $D$ be a tangle in $C'$. We remark that $D$ contains a single arc and possibly several closed components. By induction on the number of crossings of $D$, we can write $D$ via the skein relations as a scalar linear combination of tangles in $C'$, say, $D = \alpha_1 D_1 + \cdots + \alpha_k D_k$, in such a way that the single arc of any tangle $D_i$ is totally descending and that this arc lies completely below the closed components of $D_i$. Hence, $D = \gamma_1 a^{j_1} + \cdots + \gamma_k a^{j_k}$ where $\gamma_1, \ldots, \gamma_k$ are scalar multiples of tangles in $C$ and $j_1, \ldots, j_k$ are integers.

The commutativity of $C'$ now follows from the observation that terms of the kind $\gamma a^j$ commute with each other where $\gamma \in C$ and $j \in \mathbb{Z}$.

For any integer $n \geq 1$ we have a linear map $\Delta'_n: R^n \to C'$ as shown in Figure 8. This wiring $\Delta'_n$ is defined as follows. For a tangle $D$ in $R^n$, the top right boundary point of $D$ is joined with the inner boundary point of $C'$ and the bottom left boundary point of $D$ is joined with the outer boundary point of $C'$. The remaining points on the top of $D$ are joined with the remaining points on the bottom of $D$ without crossings.

With this notation, the tangle $a$ from Figure 7 is the image of the identity braid $1_z$ of $R^2$ under the map $\Delta'_2$. We abbreviate $\Delta'_n$ by $\Delta'$ if the index $n$ is obvious from the context.

We denote the image of $R^n$ under the map $\Delta'_n$ by $C'_n$ which is a submodule of $C'$. In particular, $C'_1$ is the submodule of scalar multiples of the identity $e$. We have $C'_n \cdot C'_m \subset C'_{n+m-1}$ for any integers $n, m \geq 1$. 

![Figure 6. The identity $e$ in $C'$.](image)

![Figure 7. The arc $a$ (on the left) and its inverse $a^{-1}$ (on the right).](image)
By $C_+^i$ we denote the submodule of $C'$ spanned by the set of $C_+^i$ for all $i \geq 1$. We remark that $\Delta_0'$ and $C_0'$ are not defined.

Remark. The proof of Lemma 7.1 suggests that $C_+^i$ is the polynomial ring in $a$ with coefficients in $C_+^i$ on the left (or on the right). This turns out to be true. One has to check that the powers of $a$ are linearly independent over the scalars $C_+$ (see [4] and [7, lemma 3-3-3] for two different methods).

We define a closure operation $r \mapsto \hat{r}$ from $C'$ to $C$ by putting the arc in Figure 9 over a tangle $r$. To make this possible, the annulus for $C$ has to be slightly larger than the annulus for $C'$. We remark that this closure operation is not a ring homomorphism.

The encirclement map $\Gamma: C \rightarrow C$ from Section 5 can be expressed in our new terminology as $\Gamma(f) = (f \cdot e)^\wedge$ for any $f \in C$ where $e$ is the identity element of $C'$. This simply means that adding an arc under $f$ and then adding an arc over $f$ is the same as putting a loop around $f$. This will be of importance in Section 10.

8. Basic skein relations in $C'$

We recall from Section 4 that the quasi-idempotent $a_i \in R_i^i$ satisfies $a_i a_i = \alpha_i a_i$ for a non-zero scalar $\alpha_i$ for any integer $i \geq 0$. In Section 5 we defined $h_i = (1/\alpha_i)\Delta(a_i) \in C_i$ for any integer $i \geq 0$. We defined $h_i = 0 \in C_i$ for any integer $i \leq -1$.

We now define analogue elements in the variant skein $C'$. We define $h_i' = (1/\alpha_i)\Delta'(a_i) \in C_i'$ for any integer $i \geq 1$. We define $h_i' = 0 \in C_i'$ for any integer $i \leq 0$. Recall that $C_0'$ is not defined. The following lemma is depicted in Figure 10.

**Lemma 8.1 ([10]).** We have

\[ [i + 1]h_{i+1}' = eh_i + s^{-1}[i]h_i'a \]

in $C_i'$ for any integer $i$. 

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\[ a_{i+1} = (a_i \otimes 1_1)(1_{i+1} + s\sigma_i + s^2\sigma_i\sigma_{i-1} + \cdots + s^i\sigma_i\sigma_{i-1} \cdots \sigma_1) \]

in \( R_{i+1} \) where the tensor product \( \otimes \) denotes the juxtaposition of braids. Each braid product \( \sigma_i\sigma_{i-1} \cdots \sigma_k \) where \( 2 \leq k \leq i \) can be moved anti-clockwise in the annulus to get on top of \( a_i \otimes 1 \) where it is swallowed at the expense of a scalar. It follows that \( \Delta'(a_{i+1}) \) is a linear combination of the two skein elements \( \Delta'((a_i \otimes 1_1)(\sigma_i\sigma_{i-1} \cdots \sigma_1)) \) and \( \Delta'(a_i \otimes 1_1) \) which are equal to \( e\Delta(a_i) \) and \( \Delta'(a_i) a \), respectively. Hence, \( h'_{i+1} \) is a linear combination of \( eh_i \) and \( h'a \). The scalars in this linear combination are computed in [10] and [7, lemma 3-4-1].

We define

\[ t_i = h_ie - eh_i \in C'_+ \]

for any integer \( i \). We remark that \( t_i = 0 \) for any \( i \leq 0 \).

**Lemma 8-2.** We have

\[ t_i = (s^{-i} - s^i)h_i'a \]

in \( C'_+ \) for any integer \( i \).

**Proof.** We have

\[ [i+1]h'_{i+1} = eh_i + s^{-1}[i]h'_ia \]

by Lemma 8-1 for any integer \( i \). By applying the map \( \rho \) from Section 2 we get

\[ [i+1]h'_{i+1} = h_i e + s[i]h_i'a \]

because \( h_i, h'_i \) and \( h'_{i+1} \) are invariant under \( \rho \) by Lemma 4-2. These two equations imply that \( t_i = h_i e - eh_i = (s^{-1}[i] - s[i])h_i'a \) which is equivalent to our claim.

The closure of \( t_i \) is a scalar multiple of \( h_i \).

**Lemma 8-3.** We have

\[ \hat{t}_i = (s^{1-2i} - s)vh_i \]

in \( C_+ \) for any integer \( i \).
Proof. The skein relation ($\Delta'(a_i)a)^\wedge = vs^{1-i}a_i$ depicted in Figure 11 is a consequence of Lemma 4.1. By this skein relation we deduce from Lemma 8.2 that

$$\hat{t}_i = (s^{-i} - s^i)(h'_i a)^\wedge
= (s^{-i} - s^i) vs^{1-i} h_i
= (s^{1-2i} - s)vh_i$$

for any integer $i \geq 0$. This equation holds for negative integers $i$ as well because $h_i$ and $t_i$ are equal to zero in this case.

In what follows we consider determinants. We understand a determinant in a commutative ring to be defined via the Leibniz rule.

**Lemma 8.4.** We have

$$\begin{vmatrix} h_i e & t_{i+1} \\ h_j e & t_{j+1} \end{vmatrix} = s^2 \begin{vmatrix} eh_i & t_{i+1} \\ eh_j & t_{j+1} \end{vmatrix}$$

in $C'_+$ for any integers $i$ and $j$.

Proof. We substitute in Lemma 8.1 the term $h'_i a$ by $(s^{-i} - s^i)^{-1}(h_i e - eh_i)$ using Lemma 8.2 and get

$$(s^{-i-1} - s^{i+1})h'_{i+1} = s^{-1}(h_i e) - s(eh_i).$$

In the above equation we multiply the right-hand side by $t_{j+1}$ and the left-hand side by $(s^{-j-1} - s^{j+1})h'_{j+1} a$ (which is equal to $t_{j+1}$ by Lemma 8.2) and get

$$(s^{-i-1} - s^{i+1})(s^{-j-1} - s^{j+1})h'_{i+1} h'_{j+1} a = (s^{-1}(h_i e) - s(eh_i))t_{j+1}.$$  

The left-hand side of the above equation is invariant under the interchange of $i$ and $j$ because $C'$ is commutative, and thus the right-hand side is invariant under this interchange. Hence,

$$(s^{-1}(h_i e) - s(eh_i))t_{j+1} = (s^{-1}(h_j e) - s(eh_j))t_{i+1}.$$  

This is equivalent to our claim

$$(h_i e)t_{j+1} - (h_j e)t_{i+1} = s^2(eh_i)t_{j+1} - s^2(eh_j)t_{i+1}.$$  

Lemma 8.4 is the stepping stone from skein calculations to the following algebraic calculations.
9. Determinantal calculations in \( C' \)

**Lemma 9.1.** For any integer \( r \geq 1 \) and any integers \( i_1, i_2, \ldots, i_r \), we have an equality of \((r \times r)\)-determinants in \( C'_+ \)

\[
\begin{vmatrix}
    h_{i_1} e & \cdots & h_{i_1 + r - 2} e & t_{i_1 + r - 1} \\
    \vdots & \ddots & \vdots & \vdots \\
    h_{i_r} e & \cdots & h_{i_r + r - 2} e & t_{i_r + r - 1}
\end{vmatrix}
= s^{2(r-1)}
\begin{vmatrix}
    eh_{i_1} & \cdots & eh_{i_1 + r - 2} & t_{i_1 + r - 1} \\
    \vdots & \ddots & \vdots & \vdots \\
    eh_{i_r} & \cdots & eh_{i_r + r - 2} & t_{i_r + r - 1}
\end{vmatrix}
\]

**Proof.** For \( r = 1 \) we understand the lemma to be read as \( \det(t_{i_1}) = s^{2(1-1)} \det(t_{i_1}) \) which is trivially true. The case \( r = 2 \) was proved in Lemma 8.4. From now on let \( r \geq 3 \).

Since \( t_i = h_i e - eh_i \), we deduce from Lemma 8.4 by the multilinearity of a determinant that

\[
\begin{vmatrix}
    t_{i_1} & t_{i_1 + 1} \\
    t_{j} & t_{j+1}
\end{vmatrix}
= (s^2 - 1)
\begin{vmatrix}
    eh_{i_1} & t_{i_1 + 1} \\
    eh_{j} & t_{j+1}
\end{vmatrix}
\]

(9.1)

and

\[
\begin{vmatrix}
    t_{i_1} & t_{i_1 + 1} \\
    t_{j} & t_{j+1}
\end{vmatrix}
= (1 - s^{-2})
\begin{vmatrix}
    h_{i_1} e & t_{i_1 + 1} \\
    h_{j} e & t_{j+1}
\end{vmatrix}
\]

(9.2)

We see that

\[
\begin{vmatrix}
    t_{i_1} & t_{i_1 + 1} & t_{i_1 + 2} & \cdots & t_{i_1 + r - 1} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    t_{i_r} & t_{i_r + 1} & t_{i_r + 2} & \cdots & t_{i_r + r - 1}
\end{vmatrix}
= (s^2 - 1)
\begin{vmatrix}
    eh_{i_1} & t_{i_1 + 1} & t_{i_1 + 2} & \cdots & t_{i_1 + r - 1} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    eh_{i_r} & t_{i_r + 1} & t_{i_r + 2} & \cdots & t_{i_r + r - 1}
\end{vmatrix}
\]

by developing the determinant on the left-hand side by the first two columns, applying equation (9.1) to each of the \( \binom{r}{2} \) resulting summands, and redeveloping the determinant. By doing this successively for the columns 1 and 2, then 2 and 3, \ldots, and finally for the columns \((r - 1)\) and \( r \), we deduce that

\[
\begin{vmatrix}
    t_{i_1} & \cdots & t_{i_1 + r - 2} & t_{i_1 + r - 1} \\
    \vdots & \ddots & \vdots & \vdots \\
    t_{i_r} & \cdots & t_{i_r + r - 2} & t_{i_r + r - 1}
\end{vmatrix}
= (s^2 - 1)^{r-1}
\begin{vmatrix}
    eh_{i_1} & \cdots & eh_{i_1 + r - 2} & t_{i_1 + r - 1} \\
    \vdots & \ddots & \vdots & \vdots \\
    eh_{i_r} & \cdots & eh_{i_r + r - 2} & t_{i_r + r - 1}
\end{vmatrix}
\]

(9.3)

On the other hand, if we use equation (9.2) instead of equation (9.1) in the above argument, we get

\[
\begin{vmatrix}
    t_{i_1} & \cdots & t_{i_1 + r - 2} & t_{i_1 + r - 1} \\
    \vdots & \ddots & \vdots & \vdots \\
    t_{i_r} & \cdots & t_{i_r + r - 2} & t_{i_r + r - 1}
\end{vmatrix}
= (1 - s^{-2})^{r-1}
\begin{vmatrix}
    h_{i_1} e & \cdots & h_{i_1 + r - 2} e & t_{i_1 + r - 1} \\
    \vdots & \ddots & \vdots & \vdots \\
    h_{i_r} e & \cdots & h_{i_r + r - 2} e & t_{i_r + r - 1}
\end{vmatrix}
\]

The above two equations imply that

\[
\begin{vmatrix}
    h_{i_1} e & \cdots & h_{i_1 + r - 2} e & t_{i_1 + r - 1} \\
    \vdots & \ddots & \vdots & \vdots \\
    h_{i_r} e & \cdots & h_{i_r + r - 2} e & t_{i_r + r - 1}
\end{vmatrix}
= s^{2(r-1)}
\begin{vmatrix}
    eh_{i_1} & \cdots & eh_{i_1 + r - 2} & t_{i_1 + r - 1} \\
    \vdots & \ddots & \vdots & \vdots \\
    eh_{i_r} & \cdots & eh_{i_r + r - 2} & t_{i_r + r - 1}
\end{vmatrix}
\]

We shall prove now that the skein element \( S_\lambda \in C \) as defined in Section 6 is an eigenvector of the encirclement map \( \Gamma \) from Section 5. Furthermore, the eigenvalues for \( S_\lambda \) and \( Q_\lambda \) are equal.
LEMMA 9.2. For any Young diagram $\lambda$ we have $\Gamma(S_{\lambda}) = q_{\lambda} S_{\lambda}$ in $\mathcal{C}_+$ with the scalar

$$q_{\lambda} = \frac{v^{-1} - v}{s - s^{-1}} + vs^{-1} \sum_{k=1}^{l(\lambda)} (s^{2(k-\lambda_i)} - s^{2k}).$$

Proof. We first consider the empty Young diagram. The corresponding skein element $S_{(0)}$ is the empty tangle in the annulus. Hence, $\Gamma(S_{(0)})$ is a null-homotopic, simple closed curve in the annulus which can be removed via the skein relations at the expense of multiplication with the scalar $\delta$. Hence, $\Gamma(S_{(0)}) = ((v^{-1} - v)/(s - s^{-1})) S_{(0)}$ as claimed.

In what follows we fix a non-empty Young diagram $\lambda$ and abbreviate its length $l(\lambda)$ by $r$. For any elements $\alpha$ and $\beta$ of the skein of the annulus $\mathcal{C}$ we have $(\alpha e) \cdot (\beta e) = (\alpha \beta) e$ in $\mathcal{C}'$ where $e$ is the identity of $\mathcal{C}'$. By this equality we deduce from equation (6.1) that in $\mathcal{C}'$

$$S_{\lambda} e = \det(h_{\lambda_i+j-i} e)_{1 \leq i,j \leq r}.$$  

Similarly,

$$e S_{\lambda} = \det(e h_{\lambda_i+j-i})_{1 \leq i,j \leq r}.$$  

We denote the element $S_{\lambda} e - e S_{\lambda} \in \mathcal{C}_-'$ by $\Omega_{\lambda}$. We remark that $\Omega_{(i)} = t_i$ for any integer $i \geq 0$.

By the multilinearity of a determinant we can write the difference of any two $(r \times r)$-determinants as a telescope sum of $r$ determinants of size $(r \times r)$,

$$\begin{vmatrix} y_{11} & \ldots & y_{1r} \\ \vdots & \ddots & \vdots \\ y_{r1} & \ldots & y_{rr} \end{vmatrix} - \begin{vmatrix} z_{11} & \ldots & z_{1r} \\ \vdots & \ddots & \vdots \\ z_{r1} & \ldots & z_{rr} \end{vmatrix} = \sum_{j=1}^{r} \begin{vmatrix} y_{11} & \ldots & y_{1j-1} & t_{ij} & y_{1j+1} & \ldots & y_{1r} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ y_{r1} & \ldots & y_{rj-1} & t_{rij} & z_{rj+1} & \ldots & z_{rr} \end{vmatrix}$$

where $t_{ijk} = y_{ij} - z_{ij}$. We apply this formula to the determinants for $S_{\lambda} e$ and $e S_{\lambda}$ and get

$$\Omega_{\lambda} = S_{\lambda} e - e S_{\lambda} = \det(h_{\lambda_i+j-i} e)_{1 \leq i,j \leq r} - \det(e h_{\lambda_i+j-i})_{1 \leq i,j \leq r} = \sum_{j=1}^{r} \begin{vmatrix} h_{\lambda_i} e & \ldots & h_{\lambda_i+j-2} e & t_{\lambda_i+j-1} e h_{\lambda_i+j} & \ldots & e h_{\lambda_i+r-1} e \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_i+r-1} e & \ldots & h_{\lambda_i+r-1-i} e & t_{\lambda_i+r-1-j} e h_{\lambda_i+r-1} & \ldots & e h_{\lambda_i} \end{vmatrix}$$

since $t_i = h_{ih} - e h_i$ by definition. In the above sum, we develop the $j$th summand by the first $j$ columns, apply Lemma 9.1 to each of the $(^j)$ resulting summands, and redevelop the determinant. We thus get

$$\Omega_{\lambda} = \sum_{j=1}^{r} s^{2j-2} \begin{vmatrix} e h_{\lambda_i} & \ldots & e h_{\lambda_i+j-2} & t_{\lambda_i+j-1} e h_{\lambda_i+j} & \ldots & e h_{\lambda_i+r-1} e \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ e h_{\lambda_i+r-1-j} & \ldots & e h_{\lambda_i+r-1-i} & t_{\lambda_i+r-1-j} e h_{\lambda_i+r-1} & \ldots & e h_{\lambda_i} \end{vmatrix}.$$  

The $r$ determinants appearing in this sum are very special because each of them can be written in $\mathcal{C}_-'$ by the Leibniz rule as a sum of terms of the form of a $t_i$ above a product in the $h_k$. Therefore, the closure of each of these summands is $\hat{t}_i$ multiplied
with a product in the $h_k$. Explicitly, we can write the closure of $\Omega_\lambda$ in $C_+$ as

$$\hat{\Omega}_\lambda = \sum_{j=1}^r s^{2j-2} \begin{vmatrix} h_{\lambda_1} & \cdots & h_{\lambda_1+j-2} & \hat{t}_{\lambda_1+j-1} & h_{\lambda_1+j} & \cdots & h_{\lambda_1+r-1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ h_{\lambda_r+1} & \cdots & h_{\lambda_r+j-1} & \hat{t}_{\lambda_r+j} & h_{\lambda_r+j+1} & \cdots & h_{\lambda_r} \end{vmatrix}. $$

We know by Lemma 8.3 that $\hat{t}_i$ is a scalar multiple of $h_i$. Hence,

$$\hat{\Omega}_\lambda = \sum_{j=1}^r s^{2j-2} \begin{vmatrix} h_{\lambda_1} & \cdots & h_{\lambda_1+j-2} & \beta_{ij} h_{\lambda_1+j} & h_{\lambda_1+j} & \cdots & h_{\lambda_1+r-1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ h_{\lambda_r+1} & \cdots & h_{\lambda_r+j-1} & \beta_{ij} h_{\lambda_r+j} & h_{\lambda_r+j+1} & \cdots & h_{\lambda_r} \end{vmatrix}$$

where $\beta_{ij} = s^{2j-2}(s^{1-2(\lambda_1+j-i)} - s)v$.

If $\beta_{ij}$ depended only on $j$ then $\hat{\Omega}_\lambda$ would be a scalar multiple of $S_\lambda$ because of the multilinearity of a determinant. But even in our case we can extract $\beta_{ij}$ step by step. This is due to the fact that $\beta_{ij}$ can be written as the sum of two terms that depend only on $i$ and $j$, respectively. Explicitly, we have $\beta_{ij} = \pi_i + \gamma_j$ where $\pi_i = s^{2j-2}h_i$ and $\gamma_j = -s^{2j-1}v$. The terms $\gamma_j$ can be handled by the multilinearity of a determinant in its columns. For the remaining summands we need the formula

$$\sum_{j=1}^r s^{2j-2} \begin{vmatrix} w_{11} & \cdots & w_{1j-1} & \pi_i w_{1j} & w_{1j+1} & \cdots & w_{1r} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ w_{r1} & \cdots & w_{rj-1} & \pi_r w_{rj} & w_{rj+1} & \cdots & w_{rr} \end{vmatrix} = (\pi_1 + \cdots + \pi_r) \begin{vmatrix} w_{11} & \cdots & w_{1r} \\ \vdots & & \vdots \\ w_{r1} & \cdots & w_{rr} \end{vmatrix}$$

which is valid for variables $\pi_k$ and $w_{ij}$ in any commutative ring. To prove this formula, consider the determinant $\det((1 + \pi_it)w_{ij})_{1 \leq i,j \leq r}$ where $t$ is a variable. This determinant can be written by the multilinearity in the rows as $(1 + \pi_it) \cdots (1 + \pi_rt) \det(w_{ij})_{1 \leq i,j \leq r}$. On the other hand, it can be written by the multilinearity in the columns as a power series in $t$ whose coefficients are sums of determinants in the $\pi_k$ and $w_{ij}$. The claimed formula follows from comparing the coefficients of $t^1$ in both expressions. Similar formulas with other elementary symmetric functions in the $\pi_k$ as coefficients follow from the higher degree terms in $t$.

We simplify the above formula for $\hat{\Omega}_\lambda$ by this method and get

$$\hat{\Omega}_\lambda = (\pi_1 + \cdots + \pi_r)S_\lambda + (\gamma_1 + \cdots + \gamma_r)S_\lambda$$

$$= (\beta_{11} + \cdots + \beta_{rr})S_\lambda.$$ 

The closure $(S_\lambda e)^\wedge$ is equal to $\Gamma(S_\lambda)$ as we remarked at the end of Section 7. Furthermore, the closure $(eS_\lambda)^\wedge$ is equal to $S_\lambda$ with a disjoint circle which can be removed at the expense of the scalar $\delta$. Hence,

$$\Gamma(S_\lambda) = (S_\lambda e)^\wedge$$

$$= (eS_\lambda)^\wedge + (S_\lambda e)^\wedge - (eS_\lambda)^\wedge$$

$$= (eS_\lambda)^\wedge + \hat{\Omega}_\lambda$$

$$= \delta S_\lambda + (\beta_{11} + \cdots + \beta_{rr})S_\lambda$$

$$= \left(\frac{v^{-1} - v}{s - s^{-1}} + vs^{-1} \sum_{k=1}^r (s^{2(k-\lambda_\lambda)} - s^{2k})\right) S_\lambda.$$ 

**Lemma 9.3.** $S_\lambda$ is a scalar multiple of $Q_\lambda$ for any Young diagram $\lambda$. 
Proof. Let $\lambda$ be a Young diagram. The set of $Q_\mu$ for all Young diagrams $\mu$ is a basis of $C_+$ by Corollary 5·3. Their eigenvalues with respect to the map $\Gamma$ are pairwise different by Lemma 5·1. Since the eigenvalues for $Q_\lambda$ and $S_\lambda$ are equal by Lemma 9·2, we deduce that $S_\lambda$ is a scalar multiple of $Q_\lambda$. This scalar is a rational function in $v$ and $s$, and it is possibly equal to zero.

10. Proof that $S_\lambda = Q_\lambda$

We have to introduce some notation. Let $\lambda$ be a Young diagram. A standard tableau of $\lambda$ is a labelling of the cells of $\lambda$ with the integers $1, 2, \ldots, |\lambda|$ which is increasing in each row from left to right and in each column from top to bottom. The Young diagram underlying a standard tableau $t$ is denoted by $\lambda(t)$. The number of different standard tableaux for a Young diagram $\lambda$ is denoted by $d_\lambda$.

Theorem 10·1. $S_\lambda$ is equal to $Q_\lambda$ for any Young diagram $\lambda$.

Proof. When $\lambda$ is the empty Young diagram then $Q_\lambda$ and $S_\lambda$ are equal because they are both equal to the empty tangle in the annulus.

From now on, we fix an integer $n \geq 1$ and prove that $S_\lambda$ is equal to $Q_\lambda$ for any Young diagram $\lambda$ with $n$ cells. We denote the Young diagram consisting of a single cell by.

Macdonald states in [8, example I·3·11] the multiplication rule for the product of $s_\square$ and $s_\mu$,

$$s_\square s_\mu = \sum_{\mu \subseteq \eta, |\eta| = |\mu|+1} s_\eta$$

for any Young diagram $\mu$. This is a special case of the Littlewood–Richardson rule for the multiplication of Schur functions. By applying this formula repeatedly we deduce that the $n$th power of $s_\square$ is an integer linear combination of all the $s_\lambda$ where $\lambda$ has $n$ cells. Explicitly, we have

$$s^n_\square = \sum_{|\lambda|=n} d_\lambda s_\lambda$$

where $d_\lambda$ is the number of standard tableaux of $\lambda$. Therefore, we have

$$S^n_\square = \sum_{|\lambda|=n} d_\lambda S_\lambda$$  \hspace{1cm} (10·1)$$

in $C_n$. We remark that $S_\square$ and $Q_\square$ are both equal to the single core circle of the annulus with anticlockwise orientation. $S^n_\square$ consists of $n$ parallel copies of the core circle of the annulus.

We cite an explicit semi-simple decomposition of $R^n_\square$ which was observed by Blanchet [3]. The lemmas he uses for this construction are also contained in [2] as explained in [7, section 2·5]. Generalising results of Wenzl [13], Blanchet constructs elements $\alpha_{t\tau} \in R^n_\square$ (which are denoted $\alpha_{t\beta}$ in [3]) for standard tableaux $t$ and $\tau$ with $\lambda(t) = \lambda(\tau)$ and $|\lambda(t)| = n$. These elements satisfy $\alpha_{t\tau} \alpha_{s\sigma} = \delta_{t\sigma} \alpha_{t\tau}$ where $\delta_{t\sigma}$ is the Kronecker delta, i.e. the elements $\alpha_{t\tau}$ multiply like matrix units. An important observation is that the closure $\Delta(\alpha_{t\tau}) \in C_+$ is equal to zero if $t \neq \tau$, and $\Delta(\alpha_{t\tau}) = Q_{\lambda(t)}$. 

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if \( t = \tau \). Furthermore, the identity braid \( \text{id}_n \in R_n^n \) is equal to the sum of all the \( \alpha_{tt} \),

\[
\text{id}_n = \sum_{|\lambda(t)|=n} \alpha_{tt}.
\]

We apply the closure map \( \Delta: R_n^n \to C_n \) to the above equation and we get

\[
Q_n^n = \sum_{|\lambda|=n} d_{\lambda} Q_{\lambda}.
\]  

(10·2)

Since \( S_\square = Q_\square \), equations (10·1) and (10·2) imply that

\[
\sum_{|\lambda|=n} d_{\lambda}(S_{\lambda} - Q_{\lambda}) = 0.
\]

This equation implies that \( S_{\lambda} = Q_{\lambda} \) for any Young diagram \( \lambda \) with \( n \) cells because of the following three arguments. First, any \( S_{\lambda} \) differs from \( Q_{\lambda} \) by a scalar as shown in Lemma 9·3. Second, the set \( \{Q_{\lambda} \mid \lambda \text{ has } n \text{ cells}\} \) is a basis of \( C_n \) by Lemma 5·2. And third, the integers \( d_{\lambda} \) are non-zero because they are greater than or equal to 1.

Since \( n \) was any positive integer, we have \( S_{\lambda} = Q_{\lambda} \) for any non-empty Young diagram \( \lambda \). We considered the empty Young diagram at the beginning and we have thus proved that \( S_{\lambda} = Q_{\lambda} \) for any Young diagram \( \lambda \).

**Theorem 10·2.** We consider the ring of symmetric functions as an algebra over the rational functions in \( v \) and \( s \). Then, the linear map induced by \( s_{\lambda} \mapsto Q_{\lambda} \) from the algebra of symmetric functions to the subalgebra \( C_+ \) of the skein \( C \) of the annulus is an algebra isomorphism.

**Proof.** The linear map induced by \( s_{\lambda} \mapsto S_{\lambda} \) is by definition the ring homomorphism \( \phi \) induced by \( h_i \mapsto Q_{(i)} \). By Theorem 10·1 we have \( S_{\lambda} = Q_{\lambda} \). Hence the linear map induced by \( s_{\lambda} \mapsto Q_{\lambda} \) is equal to the ring homomorphism \( \phi \).

This map is injective because the Schur functions \( s_{\lambda} \) are a basis of the symmetric functions and the \( Q_{\lambda} \) are linearly independent in \( C \). The image of this map is equal to \( C_+ \) because the \( Q_{\lambda} \) are a basis of \( C_+ \) by Corollary 5·3.

From equation (3·1) and Theorem 10·1 we deduce

**Theorem 10·3.**

\[
Q_{\lambda} Q_{\mu} = \sum_{|\nu| = |\lambda|+|\mu|} a_{\lambda\mu}^{\nu} Q_{\nu}
\]

in \( C \) for any Young diagrams \( \lambda \) and \( \mu \). The non-negative integers \( a_{\lambda\mu}^{\nu} \) are the Littlewood–Richardson coefficients.

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**REFERENCES**


Papers marked with a star * are available online at the Liverpool Knot Theory archive http://www.liv.ac.uk/~su14/knotprints.html.